

ON ORBIT CLOSURES FOR INFINITE TYPE QUIVERS

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ABSTRACT. For the Kronecker quiver, Zwara [7, Theorem 1] has found an example of a representation whose orbit closure is neither unibranch nor Cohen-Macaulay. In this note, we explain how to extend this example to all infinite type quivers without oriented cycles.

In this short note, we use quiver exceptional sequences to reduce, in a "hom-controlled" manner, the list of all representation-infinite quivers without oriented cycles to just \tilde{A}_2 . This observation combined with Zwara's example from [7, Theorem 1] yields:

Theorem 0.1. *Let Q be a representation-infinite, connected quiver without oriented cycles. Then, there exists a representation W whose orbit closure is neither unibranch nor Cohen-Macaulay.*

In [6], Zwara showed that the orbit closures of representations of Dynkin quivers are always unibranch. Hence, we deduce that the Dynkin quivers are precisely the ones with the property that all orbit closures are unibranch.

In what follows, we first review some background material from quiver theory and then prove Theorem 0.1. Throughout this note, we work over an algebraically closed field k of characteristic zero.

Let $Q = (Q_0, Q_1, t, h)$ be a finite quiver, where Q_0 is the set of vertices, Q_1 is the set of arrows and $t, h : Q_1 \rightarrow Q_0$ assign to each arrow $a \in Q_1$ its tail ta and head ha , respectively. A representation V of Q over k is a family of finite dimensional k -vector spaces $\{V(x) \mid x \in Q_0\}$ together with a family $\{V(a) : V(ta) \rightarrow V(ha) \mid a \in Q_1\}$ of k -linear maps. If V is a representation of Q , we define its dimension vector \underline{d}_V by $\underline{d}_V(x) = \dim_k V(x)$ for every $x \in Q_0$. Thus the dimension vectors of representations of Q lie in $\Gamma = \mathbb{Z}^{Q_0}$, the set of all integer-valued functions on Q_0 .

Given two representations V and W of Q , we define a morphism $\varphi : V \rightarrow W$ to be a collection of linear maps $\{\varphi(x) : V(x) \rightarrow W(x) \mid x \in Q_0\}$ such that for every arrow $a \in Q_1$, we have $\varphi(ha)V(a) = W(a)\varphi(ta)$. We denote by $\text{Hom}_Q(V, W)$ the k -vector space of all morphisms from V to W . Let W and V be two representations of Q . We say that V is a subrepresentation of W if $V(x)$ is a subspace of $W(x)$ for all vertices $x \in Q_0$ and $V(a)$ is the restriction of $W(a)$ to $V(ta)$ for all arrows $a \in Q_1$. In this way, we obtain the abelian category $\text{Rep}(Q)$ of all quiver representations of Q .

A representation W is said to be a *Schur representation* if $\text{End}_Q(W) \cong \mathbb{C}$. The dimension vector of a Schur representation is called a *Schur root*.

From now on, we assume that our quivers are without oriented cycles. For two quiver representations V and W , consider Ringel's canonical exact sequence [5] :

$$(1) \quad 0 \rightarrow \text{Hom}_Q(V, W) \rightarrow \bigoplus_{x \in Q_0} \text{Hom}_k(V(x), W(x)) \xrightarrow{d_W^V} \bigoplus_{a \in Q_1} \text{Hom}_k(V(ta), W(ha)),$$

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where $d_W^V((\varphi(x))_{x \in Q_0}) = (\varphi(ha)V(a) - W(a)\varphi(ta))_{a \in Q_1}$ and $\text{Ext}_Q^1(V, W) = \text{coker}(d_W^V)$.

If α, β are two elements of Γ , we define the Euler inner product by

$$(2) \quad \langle \alpha, \beta \rangle_Q = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

(When no confusion arises, we drop the subscript Q .)

It follows from (1) and (2) that

$$\langle \underline{d}_V, \underline{d}_W \rangle = \dim_k \text{Hom}_Q(V, W) - \dim_k \text{Ext}_Q^1(V, W).$$

A dimension vector β is called a *real Schur root* if there exists a representation $W \in \text{Rep}(Q, \beta)$ such that $\text{End}_Q(W) \simeq k$ and $\text{Ext}_Q^1(W, W) = 0$ (we call such a representation *exceptional*). Note that if β is a real Schur root then there exists a unique, up to isomorphism, exceptional β -dimensional representation.

For α and β two dimension vectors, consider the generic ext and hom:

$$\text{ext}_Q(\alpha, \beta) := \min\{\dim_k \text{Ext}_Q^1(V, W) \mid (V, W) \in \text{Rep}(Q, \alpha) \times \text{Rep}(Q, \beta)\},$$

and

$$\text{hom}_Q(\alpha, \beta) := \min\{\dim_k \text{Hom}_Q(V, W) \mid (V, W) \in \text{Rep}(Q, \alpha) \times \text{Rep}(Q, \beta)\}.$$

Given two dimension vectors α and β , we write $\alpha \perp \beta$ provided that $\text{ext}_Q(\alpha, \beta) = \text{hom}_Q(\alpha, \beta) = 0$.

Definition 0.2. We say that $(\varepsilon_1, \dots, \varepsilon_r)$ is an *exceptional sequence* if

- (1) the ε_i are real Schur roots;
- (2) $\varepsilon_i \perp \varepsilon_j$ for all $1 \leq i < j \leq l$.

Following [4], a sequence $(\varepsilon_1, \dots, \varepsilon_r)$ is called a *quiver exceptional sequence* if it is exceptional and $\langle \varepsilon_j, \varepsilon_i \rangle \leq 0$ for all $1 \leq i < j \leq l$.

Now, let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$ be a quiver exceptional sequence and let $E_i \in \text{Rep}(Q, \varepsilon_i)$ be exceptional representations. Construct a new quiver $Q(\varepsilon)$ with vertex set $\{1, \dots, r\}$ and $-\langle \varepsilon_j, \varepsilon_i \rangle$ arrows from j to i . Define $\mathcal{C}(\varepsilon)$ to be the smallest full subcategory of $\text{Rep}(Q)$ which contains E_1, \dots, E_r and is closed under extensions, kernels of epimorphisms, and cokernels of monomorphisms.

For the remaining of this section, we assume that $r \leq N - 1$, where N is the number of vertices of Q . We recall a very useful result from [4, Section 2.7] in a form that is convenient for us (see also [3]):

Proposition 0.3. [4] *The category $\mathcal{C}(\varepsilon)$ is naturally equivalent to $\text{Rep}(Q(\varepsilon))$ with E_1, \dots, E_r being the simple objects of $\mathcal{C}(\varepsilon)$. Furthermore, the inverse functor from $\text{Rep}(Q(\varepsilon))$ to $\mathcal{C}(\varepsilon)$ is a full exact embedding into $\text{Rep}(Q)$.*

Consider the linear transformation

$$I : \mathbb{Z}^{Q(\varepsilon)_0} \rightarrow \mathbb{Z}^{Q_0}$$

$$\alpha = (\alpha(1), \dots, \alpha(r)) \rightarrow \sum_{i=1}^r \alpha(i)\varepsilon_i,$$

If $F : \text{Rep}(Q(\varepsilon)) \rightarrow \text{Rep}(Q)$ is the full exact embedding from Proposition 0.4 then F is clearly hom-controlled in the sense of Zwara [8]. Let

$$F^{(\alpha)} : \text{Rep}(Q(\varepsilon), \alpha) \rightarrow \text{Rep}(Q, I(\alpha))$$

be the induced morphism of varieties. Zwara [8, Theorem 2] (see also [2, Proposition 9]) showed that hom-controlled functors preserve the type of singularities of orbit closures. In particular, we have:

Proposition 0.4. *Keep the same notations as above. Let $V \in \text{Rep}(Q(\varepsilon), \alpha)$. Then, $\overline{\text{GL}(\alpha)V}$ is normal (or unibranch or Cohen-Macaulay) if and only if $\overline{\text{GL}(I(\alpha))F^{(\alpha)}(V)}$ has the same property.*

Next, let us recall Zwara's example from [7]:

Theorem 0.5. *Let $\theta(2)$ be the Kronecker quiver*

$$1 \rightrightarrows 2$$

Label the arrows by a and b . Consider the following representation $V \in \text{Rep}(\theta(2), (3, 3))$ defined by $V(a) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $V(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, $\overline{\text{GL}(\alpha)V}$ is neither unibranch nor Cohen-Macaulay.

Proof of Theorem 0.1. From [1, Lemma 2.1, pp. 253], we know that any finite, connected quiver Q of infinite representation type must contain a Euclidean quiver as a subquiver. Therefore, it is enough to prove the theorem for Euclidean quivers. Now, let Q be a Euclidean quiver and denote by δ_Q the isotropic Schur root of Q .

Choose v to be a vertex such that $Q \setminus v$ is a Dynkin quiver. Without loss of generality, let us assume that v is a source. In this case, we take $\varepsilon_1 = \delta_Q - e_v$ and $\varepsilon_2 = e_v$. Then, $(\varepsilon_1, \varepsilon_2)$ is a quiver exceptional sequence with $\langle \varepsilon_2, \varepsilon_1 \rangle = -2$ and so the proof follows from Proposition 0.4 and Theorem 0.5. \square

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